# Logistic Regression

## ACTL3142 & ACTL5110 Statistical Machine Learning for Risk Applications

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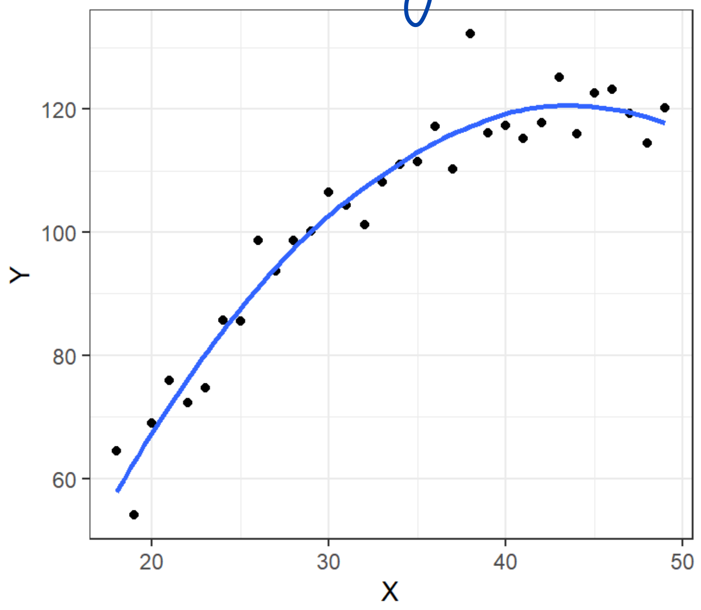
## Lecture Outline

- **An overview of classification**
- Logistic regression – Classification type problem
- Poisson regression – Counting type problems
- Generalised linear models

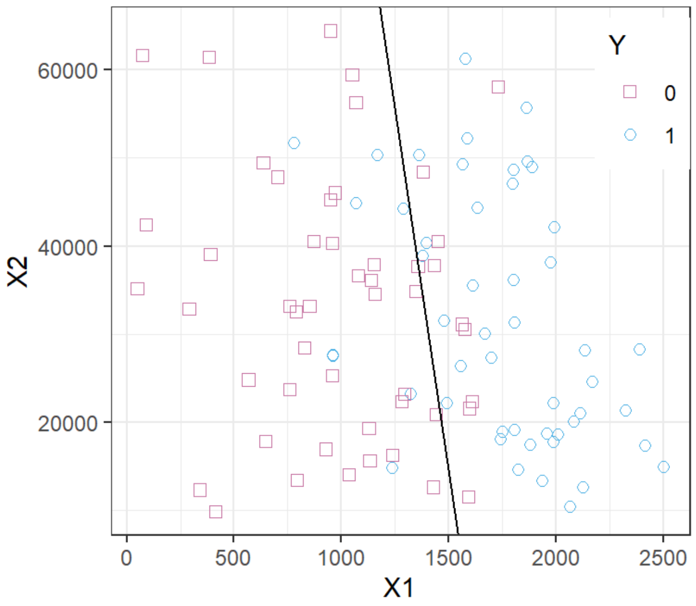
# Regression vs. classification

## Regression

linear regression




## Classification




- $Y$ is quantitative, continuous
- Examples: Sales prediction, claim size prediction, stock price modelling

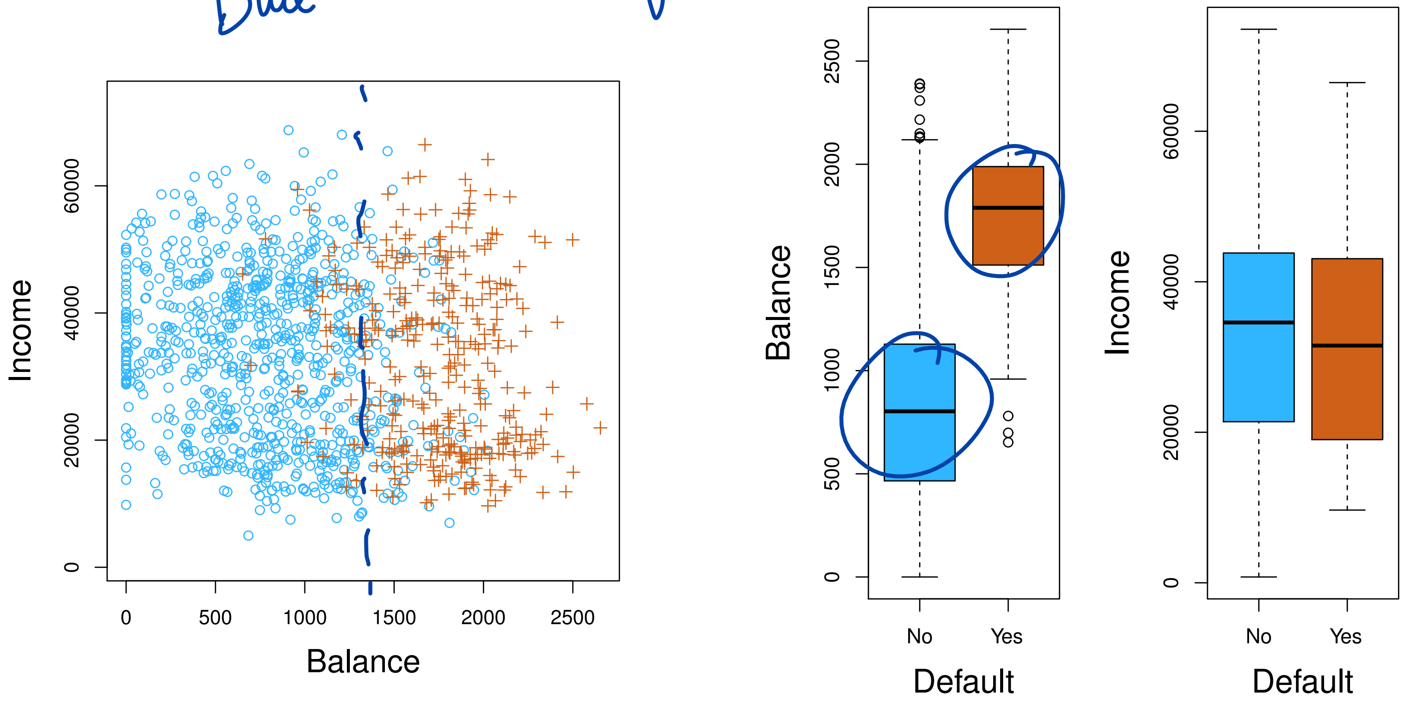
- $Y$ is qualitative, discrete
- Examples: Fraud detection, face recognition, accident occurrence, death

# Some examples of classification problems

- Success/failure of a treatment, explained by dosage of medicine administered, patient's age, sex, weight and severity of condition, etc.
- Vote for/against political party, explained by age, gender, education level, region, ethnicity, geographical location, etc.
- Customer churns/stays depending on usage pattern, complaints, social demographics, etc.

# Example: Predicting defaults (`Default` from `ISLR2`)

Blue = no default



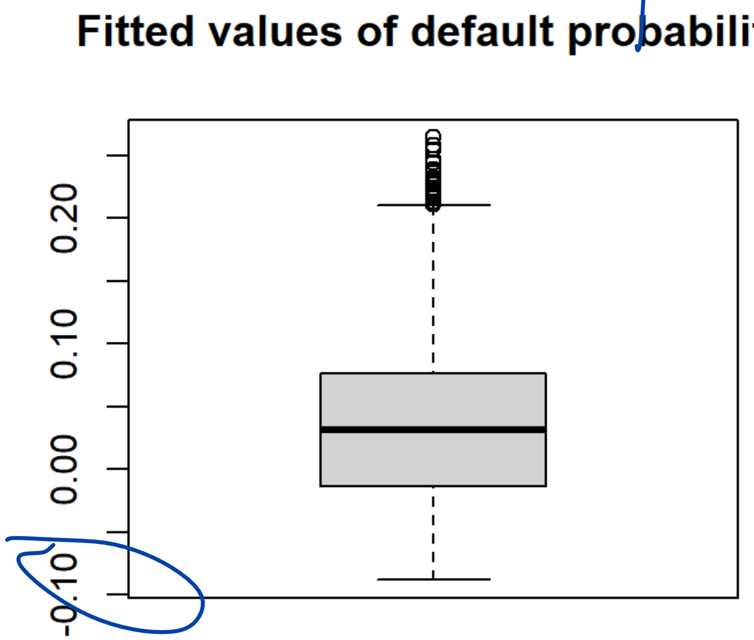

- `default` ($Y$) is a binary variable (yes/no or 0/1)
- Annual `income` ($X_1$) and credit card `balance` ($X_2$) may be continuous predictors

# Example: Predicting defaults - Discussion

Simple linear regression on `Default` data:

$Y = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p$

prob. of default.




have negative value

Non-sensical

What do you observe?

# Classification problems

- Coding in the binary case is simple

$$Y \in \{0, 1\} \Leftrightarrow Y \in \{\textcolor{blue}{\bullet}, \textcolor{green}{\bullet}\}$$

- Our objective is to find a good predictive model $f$ that can:

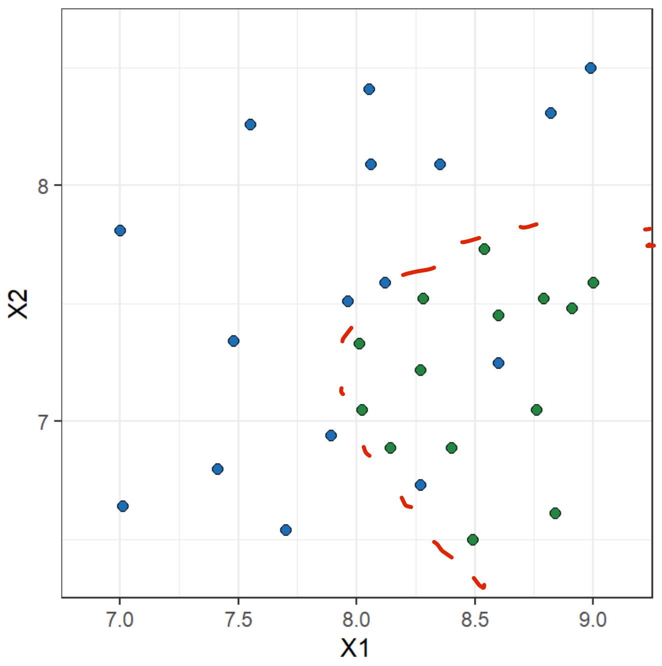
  1. Estimate the probability $\mathbb{P}(Y = 1|X) \in \{0, 1\}$

$$f(X) \rightarrow \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet$$

0 1

  2. Classify observation

$$f(X) \rightarrow \hat{Y} \in \{\textcolor{blue}{\bullet}, \textcolor{green}{\bullet}\}$$



## Lecture Outline

- An overview of classification
- **Logistic regression**
- Poisson regression
- Generalised linear models

# Logistic regression

Extend linear regression to model binary categorical variables

$$\underbrace{\ln\left(\frac{\mathbb{P}(Y=1|X)}{1-\mathbb{P}(Y=1|X)}\right)}_{\text{log-odds}} = \underbrace{\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p}_{\text{linear model}}$$

Same as MLR structure

log-odds are linear in X.

$\log\left(\frac{p}{1-p}\right)$

odds.

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# Principles of Logistic Regression

- The output is binary $Y \in \{1, 0\}$ , Code your classification as 0, 1.
- Each case's $Y$ variable has a probability between 0 and 1 that depends on the values of the predictors $X$ such that

$$\mathbb{P}(Y = 1|X) + \mathbb{P}(Y = 0|X) = 1$$

- Probability can be restated as odds

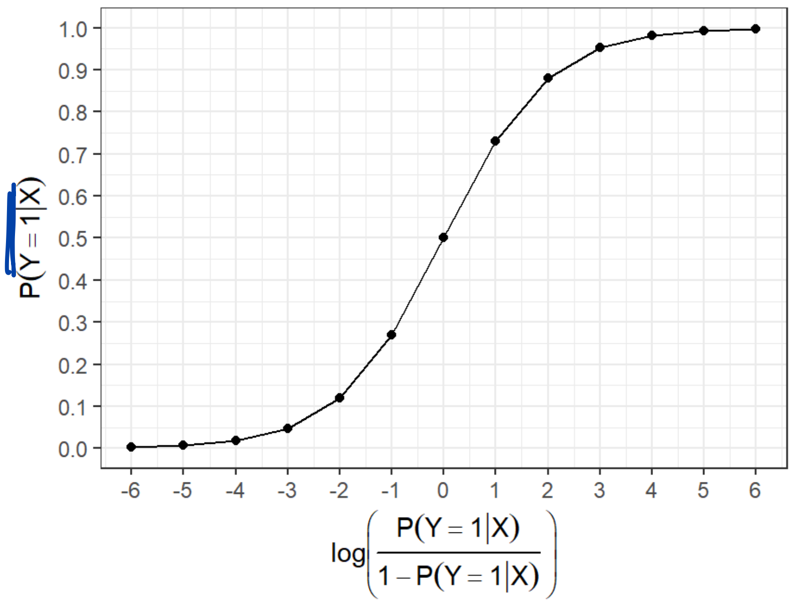
$$\text{Odds}(Y = 1|X) = \frac{\mathbb{P}(Y = 1|X)}{\mathbb{P}(Y = 0|X)} = \frac{\mathbb{P}(Y = 1|X)}{1 - \mathbb{P}(Y = 1|X)}$$

- Odds are a measure of relative probabilities

# Probabilities, odds and log-odds

**Goal:** Transform a number between 0 and 1 into a number between $-\infty$ and $-\infty$

| probability | odds | logodds |
|---|---|---|
| 0.001 | 0.001 | -6.907 |
| 0.250 | 0.333 | -1.099 |
| 0.500 | 1.000 | 0.000 |
| 0.750 | 3.000 | 1.099 |
| 0.999 | 999.000 | 6.907 |




$P(Y=1|X)=0.5$

$P(Y=0|X)=0.5$

transformation is unique

# Logistic regression

- Perform regression on log-odds

$$\ln\left(\frac{\mathbb{P}(Y=1|X)}{1-\mathbb{P}(Y=1|X)}\right) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

- Use (training) data and maximum-likelihood estimation to produce estimates $\hat{\beta}_0, \hat{\beta}_1, \ldots \hat{\beta}_p$.
- Predict probabilities using

$$\underline{\mathbb{P}(Y=1|X)} = \frac{e^{\hat{\beta}_0+\hat{\beta}_1 X_1+\cdots+\hat{\beta}_p X_p}}{1+e^{\hat{\beta}_0+\hat{\beta}_1 X_1+\cdots+\hat{\beta}_p X_p}}$$

Do you know where this comes from?

$$\ln\left(\frac{p}{1-p}\right) = X\beta$$

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# Interpretation of coefficients

$\ln\left(\frac{p}{1-p}\right) = X\beta$

log-odds

- Recall for **multiple linear regression** we model the response as

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \varepsilon.$$

An increase of the entry $x_{ij}$ by 1 in $X$ we would predict $Y_i$ to increase by $\hat{\beta}_j$ on average since

$$\mathbb{E}[Y_i|X] = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \cdots + \hat{\beta}_j(x_{ij} + 1) + \cdots + \hat{\beta}_p x_{ip}$$

- For **logistic regression** we have a similar relationship. When $x_{ij}$ increases by 1 we would expect the **log-odds** for $Y_i$ to increase by $\beta_j$.
- The new predicted probability of success by increasing $x_{ij}$ by 1 is now

$$\mathbb{P}(Y_i = 1|X) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \cdots + \hat{\beta}_j(x_{ij}+1) + \cdots + \hat{\beta}_p x_{ip}}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \cdots + \hat{\beta}_j(x_{ij}+1) + \cdots + \hat{\beta}_p x_{ip}}}.$$

Convince yourself that the probability does increase if $\beta_j$ is positive!

# How are the coefficients estimated?

- Recall the Bernoulli distribution is parameterised by a parameter $p$ and has the density

$$f(y) = p^y(1-p)^{1-y}.$$

- In logistic regression we maximise the likelihood of the data. Denote

$$p(y_i;\beta) = \frac{1}{1-\mathrm{e}^{-\mathrm{x}_i\beta}},$$

where $\mathrm{x}_i$ denotes the $i$'th row of $X$.

- We maximise the log-likelihood below

$$\ell(\beta) = \sum_{i=1}^{n} y_i \ln p(y_i;\beta) + (1-y_i)\ln(1-p(y_i;\beta)).$$

We take partials w.r.t. to each $\beta_j$ and set to 0. Needs numerical approximation.

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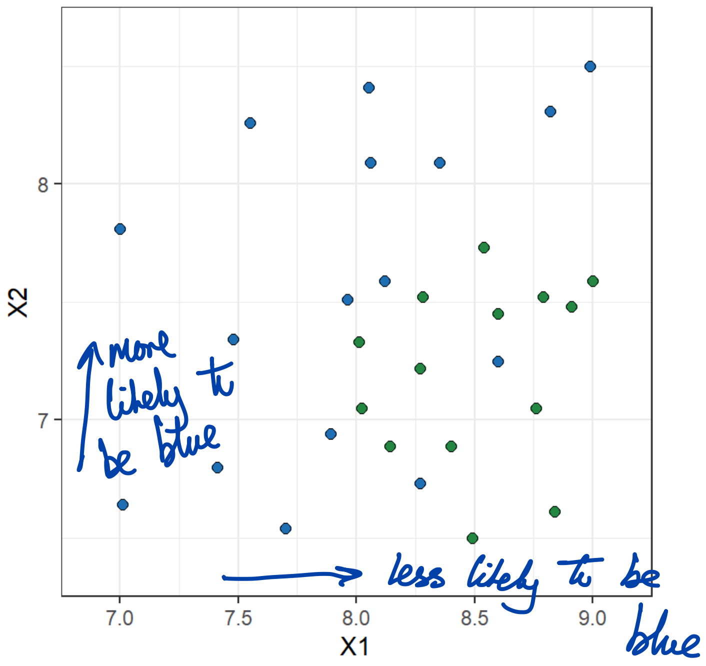
# Toy example: Logistic Regression

$$Y = \begin{cases} 1 & \text{if } \color{blue}{\bullet} \\ 0 & \text{if } \color{green}{\bullet} \end{cases} \qquad \ln\left(\frac{\mathbb{P}(Y=1|X)}{1-\mathbb{P}(Y=1|X)}\right) = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

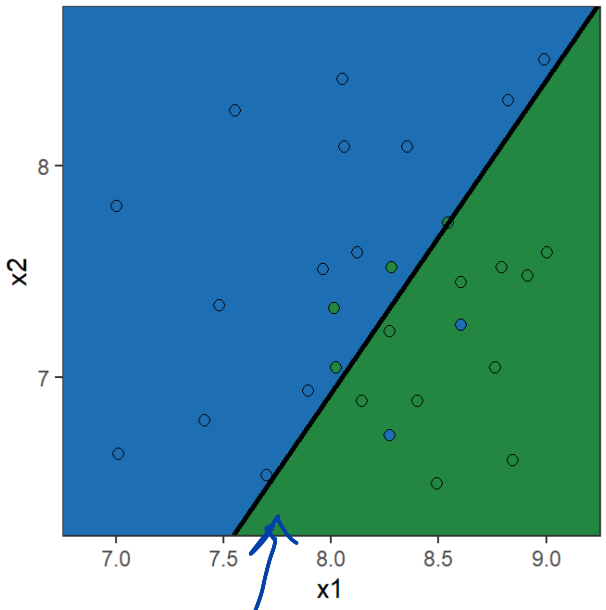
- The parameter estimates are $\hat{\beta}_0 = 13.671$, $\hat{\beta}_1 =$ -4.136, $\hat{\beta}_2 = 2.803$
- $\hat{\beta}_1 =$ -4.136 implies that the bigger $X_1$ the lower the chance it is a blue point
- $\hat{\beta}_2 = 2.803$ implies that the bigger $X_2$ the higher the chance it is a blue point



# Toy example: Logistic Regression

$$\underbrace{\ln\left(\frac{\mathbb{P}(Y=1|X)}{1-\mathbb{P}(Y=1|X)}\right)}_{\text{log odds}} = \underbrace{13.671 - 4.136X_1 + 2.803X_2}_{\text{linear model}}$$

| X1 | X2 | log-odds | P(Y=1\|X) | prediction |
|---|---|---|---|---|
| 7.0 | 8.0 | 7.14 | 0.9992 | blue |
| 8.0 | 7.5 | 1.61 | 0.8328 | blue |
| 8.0 | 7.0 | 0.20 | 0.5508 | blue |
| 8.5 | 7.5 | -0.46 | 0.3864 | green |
| 9.0 | 7.0 | -3.93 | 0.0192 | green |




linear division of the predictor space.

# Some important points about logistic regression

- Changes in predictor values correspond to changes in the *log-odds*, not the probability
- Evaluating predictors to add / remove is the same as in linear regression. The only change is the form of the response

  still use hypothesis tests (but z-tests now).
- As a result, most of the modelling limitations of linear regression (e.g. collinearity) carry over as well
- Possible to do logistic regression on non-binary responses, but not used that often, and not covered here

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# Example: Predicting defaults

```r
glmStudent <- glm(default ~ student, family = binomial(), data = ISLR2::Default)
summary(glmStudent)
```

```
Call:
glm(formula = default ~ student, family = binomial(), data = ISLR2::Default)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)    
(Intercept) -3.50413    0.07071  -49.55  < 2e-16 ***
studentYes   0.40489    0.11502    3.52 0.000431 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2920.6  on 9999  degrees of freedom
Residual deviance: 2908.7  on 9998  degrees of freedom
AIC: 2912.7

Number of Fisher Scoring iterations: 6
```

familiar

next week

If default represents Y=1, you are more likely to default if you are a students.

# Example: Predicting defaults

```
glmAll <- glm(default ~ balance + income + student, family = binomial(), data = ISLR2::Default)
summary(glmAll)
```

```
Call:
glm(formula = default ~ balance + income + student, family = binomial(),
    data = ISLR2::Default)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.087e+01  4.923e-01 -22.080  < 2e-16 ***
balance      5.737e-03  2.319e-04  24.738  < 2e-16 ***
income       3.033e-06  8.203e-06   0.370  0.71152
studentYes  -6.468e-01  2.363e-01  -2.738  0.00619 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2920.6  on 9999  degrees of freedom
Residual deviance: 1571.5  on 9996  degrees of freedom
AIC: 1579.5

Number of Fisher Scoring iterations: 8
```

In the presence of balance and income, you are less likely to default if you are a student.

# Example: Predicting defaults - Discussion

Results of logistic regression:

`default` against `student`

| Predictor | Coefficient | Std error | Z-statistic | P-value |
|---|---|---|---|---|
| `(Intercept)` | -3.5041 | 0.0707 | -49.55 | <0.0001 |
| `student = Yes` | 0.4049 | 0.1150 | 3.52 | 0.0004 |

`default` against `balance`, `income`, and `student`

| Predictor | Coefficient | Std error | Z-statistic | P-value |
|---|---|---|---|---|
| `(Intercept)` | -10.8690 | 0.4923 | -22.080 | < 0.0001 |
| `balance` | 0.0057 | 2.319e-04 | 24.738 | < 0.0001 |
| `income` | 0.0030 | 8.203e-06 | 0.370 | 0.71152 |
| `student = Yes` | -0.6468 | 0.2362 | -2.738 | 0.00619 |

# Assessing accuracy in classification problems

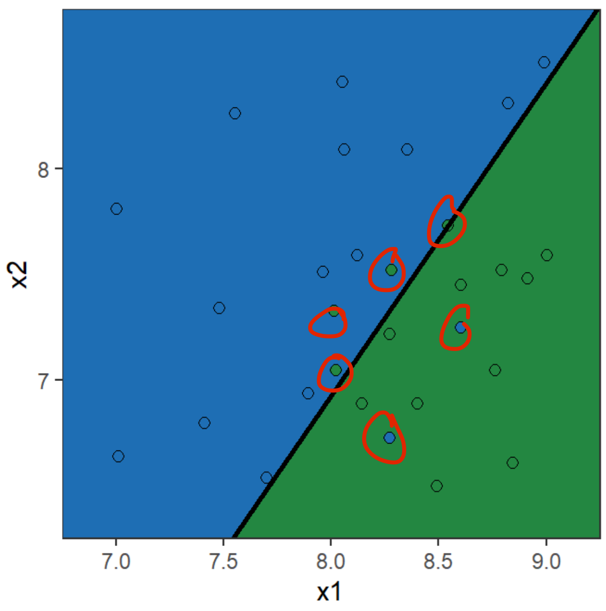
- We assess model accuracy using the error rate

$$\text{error rate} = \frac{1}{n}\sum_{i=1}^{n} \underbrace{I(y_i \neq \hat{y}_i)}_{1 \text{ if } y_i \neq \hat{y}_i}$$

- In our toy example with a 50% threshold

$$\text{training error rate} = \frac{6}{30} = 0.2$$

If $P(Y=1|X) > 0.5$, predict blue

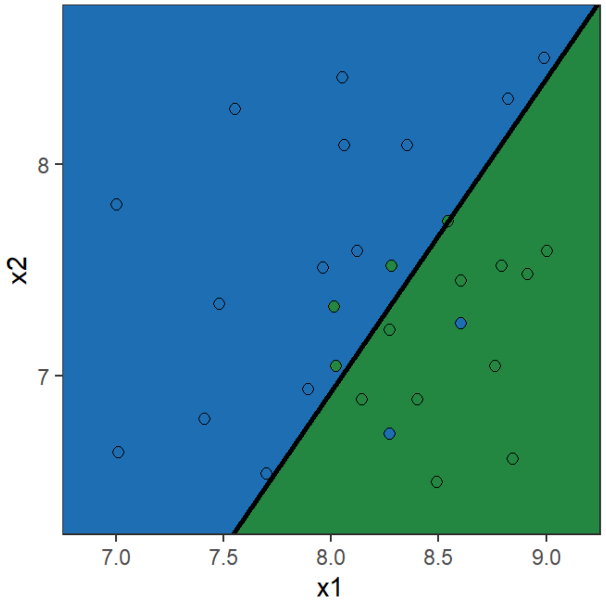


# Confusion matrix: Toy example (50% Threshold)

- Confusion matrix

| | $Y = 0$ | $Y = 1$ | **Total** |
|---|---|---|---|
| $\hat{Y} = 0$ | 10 | 2 | 12 |
| $\hat{Y} = 1$ | 4 | 14 | 18 |
| Total | 14 | 16 | 30 |

- True-Positive Rate $= \frac{14}{16} = 0.875$
- False-Positive Rate $= \frac{4}{14} = 0.286$

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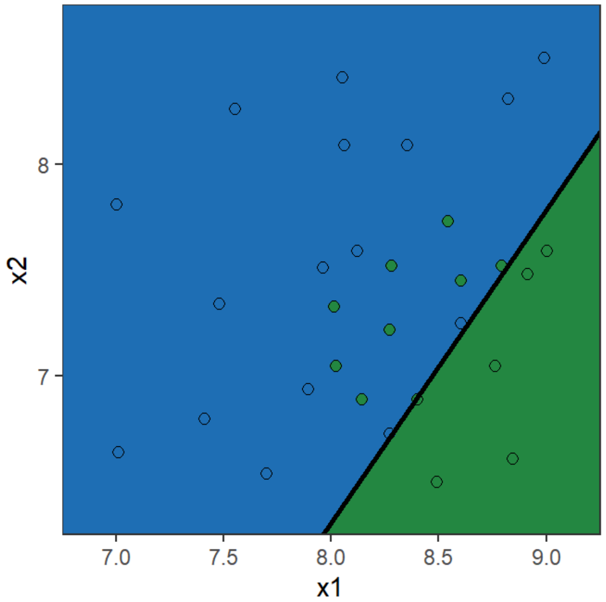
# Confusion matrix: Toy example (15% Threshold)

- Confusion matrix

| | $Y = 0$ | $Y = 1$ | **Total** |
|---|---|---|---|
| $\hat{Y} = 0$ | 6 | 0 | 6 |
| $\hat{Y} = 1$ | 8 | 16 | 24 |
| Total | 14 | 16 | 30 |

- True-Positive Rate $= \frac{16}{16} = 1$
- False-Positive Rate $= \frac{8}{14} = 0.429$

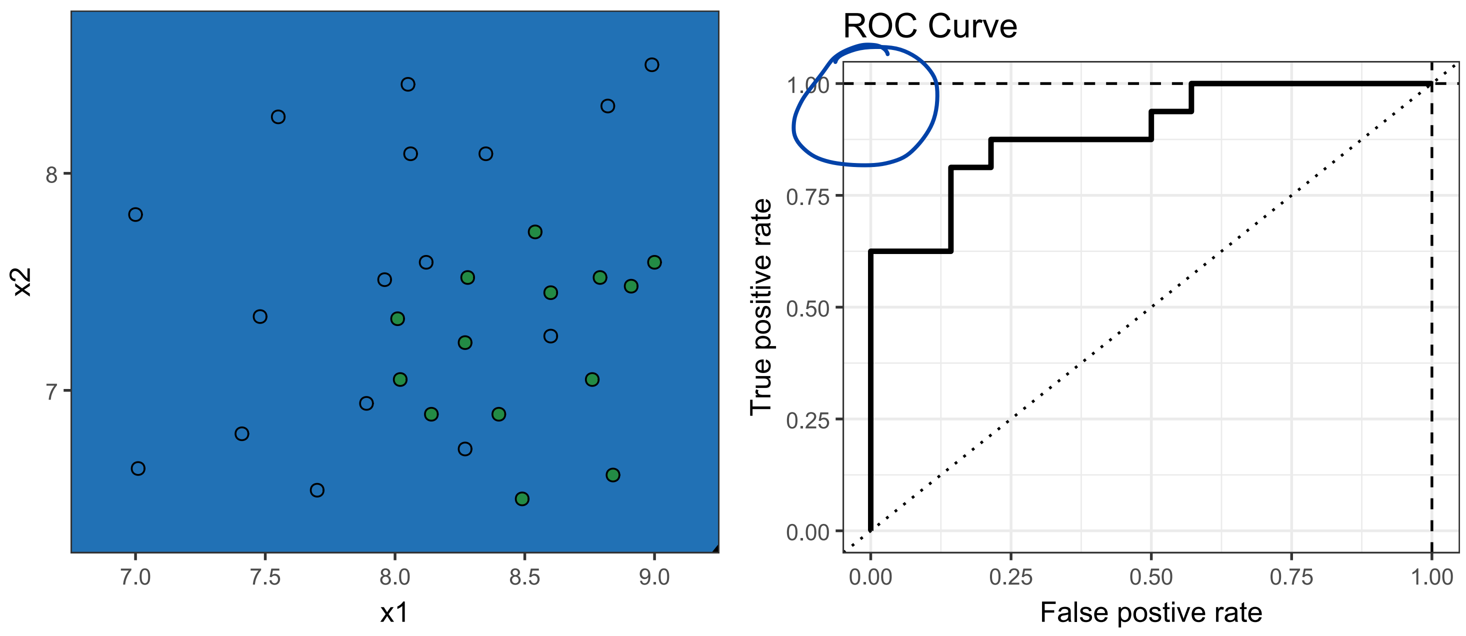
larger false positive rate




Predict 1 if

$P(Y=1|X) > 0.15$

# ROC Curve and AUC: Toy example




- ROC Curve: Plots the true-positive rate against the false-positive rate
- A good model will have its ROC curve hug the top-left corner more
- AUC is the area under the ROC curve: For this toy example AUC$=0.8929$

## Lecture Outline

- An overview of classification
- Logistic regression
- **Poisson regression**
- Generalised linear models

# Poisson regression - Motivation

In many application we need to model count data:

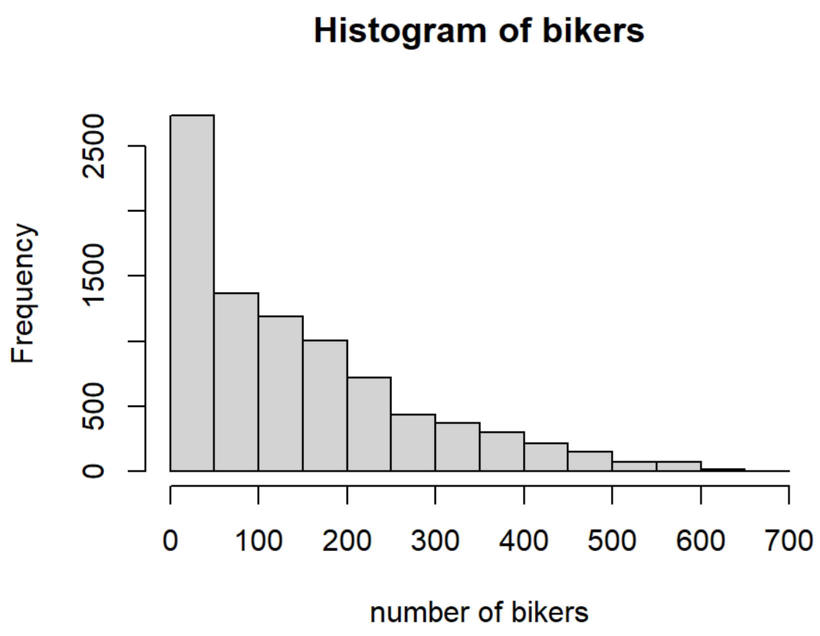
- In mortality studies the aim is to explain the number of deaths in terms of variables such as age, gender and lifestyle.
- In health insurance, we may wish to explain the number of claims made by different individuals or groups of individuals in terms of explanatory variables such as age, gender and occupation.
- In general insurance, the count of interest may be the number of claims made on vehicle insurance policies. This could be a function of the color of the car, engine capacity, previous claims experience, and so on.

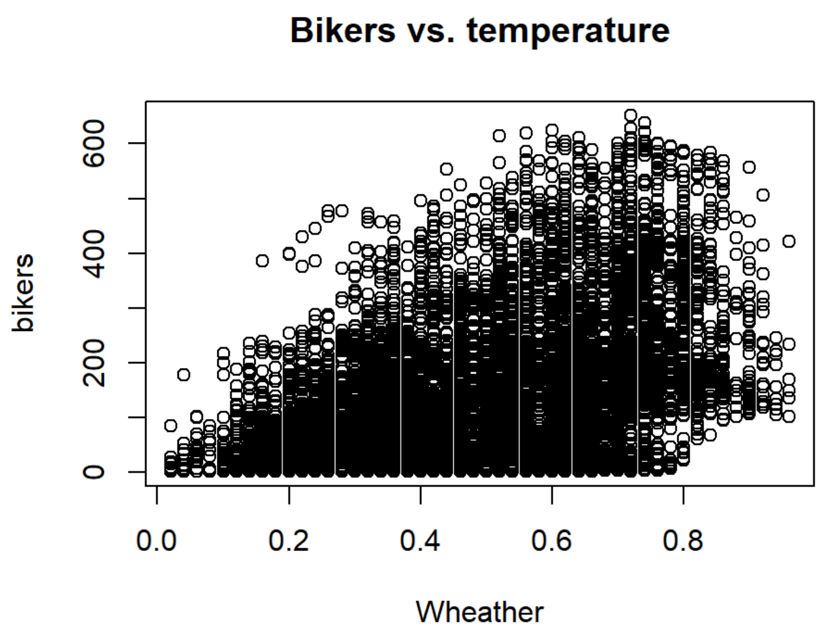
# The Bikeshare dataset

```r
str(ISLR2::Bikeshare)
```

```
'data.frame':	8645 obs. of  15 variables:
 $ season    : num  1 1 1 1 1 1 1 1 1 1 ...
 $ mnth      : Factor w/ 12 levels "Jan","Feb","March",..: 1 1 1 1 1 1 1 1 1 1 ...
 $ day       : num  1 1 1 1 1 1 1 1 1 1 ...
 $ hr        : Factor w/ 24 levels "0","1","2","3",..: 1 2 3 4 5 6 7 8 9 10 ...
 $ holiday   : num  0 0 0 0 0 0 0 0 0 0 ...
 $ weekday   : num  6 6 6 6 6 6 6 6 6 6 ...
 $ workingday: num  0 0 0 0 0 0 0 0 0 0 ...
 $ weathersit: Factor w/ 4 levels "clear","cloudy/misty",..: 1 1 1 1 1 2 1 1 1 1 ...
 $ temp      : num  0.24 0.22 0.22 0.24 0.24 0.24 0.22 0.2 0.24 0.32 ...
 $ atemp     : num  0.288 0.273 0.273 0.288 0.288 ...
 $ hum       : num  0.81 0.8 0.8 0.75 0.75 0.75 0.8 0.86 0.75 0.76 ...
 $ windspeed : num  0 0 0 0 0 0.0896 0 0 0 0 ...
 $ casual    : num  3 8 5 3 0 0 2 1 1 8 ...
 $ registered: num  13 32 27 10 1 1 0 2 7 6 ...
 $ bikers    : num  16 40 32 13 1 1 2 3 8 14 ...
```

# The `Bikeshare` dataset - Discussion




How could we model the number of `bikers` as function of the other variables?

# Why not use muliple linear regression?

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \epsilon$$

- Could predict negative values
- Constant variance may be inadequate
- Assumes continuous numbers while counts are integers

$$\log(Y) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \epsilon$$

- Solves problem of negative values
- May solve constant variance problem
- Assumes continuous numbers while counts are integers
- Not applicable with zero counts

# Poisson regression

- Assume that $Y \sim \text{Poisson}(\lambda)$

$$\mathbb{P}(Y = k) = \frac{e^{\lambda}\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \dots \quad \text{with } \mathbb{E}[Y] = \text{Var}(Y) = \lambda$$

- Assume that $\mathbb{E}[Y] = \lambda(X_1, \dots, X_p)$ is log-linear in the predictors

$$\boxed{\log(\lambda(X_1, \dots, X_p)) = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p}$$

- Use data and maximum-likelihood estimation to obtain $\hat{\beta}_0, \hat{\beta}_1, \dots \hat{\beta}_p$

$$\mathcal{L}(\beta_0, \beta_1, \dots, \beta_p) = \prod_{i=1}^{n} \frac{e^{\lambda(x_i)}\lambda(x_i)^{y_i}}{y_i!} \quad \text{with} \quad \lambda(x_i) = \beta_0 + \beta_1 x_{i1} + \dots + \beta_p x_{p1}$$

# Some important points about Poisson regression

- Interpretation: An increase in $X_j$ by one unit is associated with a change in $\mathbb{E}[Y]$ by a factor $e^{\beta_j}$.
- Mean-variance relationship: $\mathbb{E}[Y] = \text{Var}(Y) = \lambda$ implies that the variance is non-constant and increases with the mean.
- Non-negative fitted values: Predictions are always positive
- Evaluating predictors to add / remove is the same as in linear regression. The only change is the form of the response
- As a result, most of the modelling limitations of linear regression (e.g. collinearity) carry over as well

$$\log(\lambda) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p.$$

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# Poisson regression - Bikeshare dataset

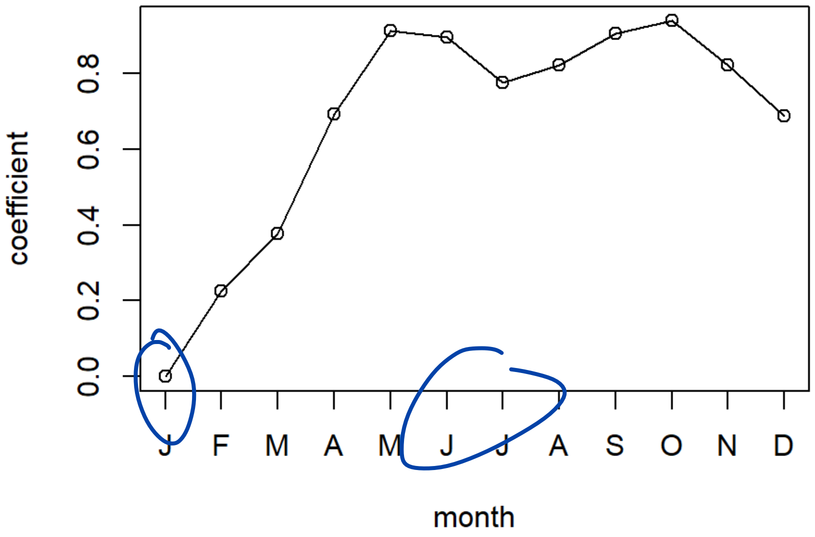
```r
glmBikeshare <- glm(bikers ~ workingday + temp + weathersit + mnth + hr,  family = poisson(),
                    data = ISLR2::Bikeshare)
summary(glmBikeshare)
```

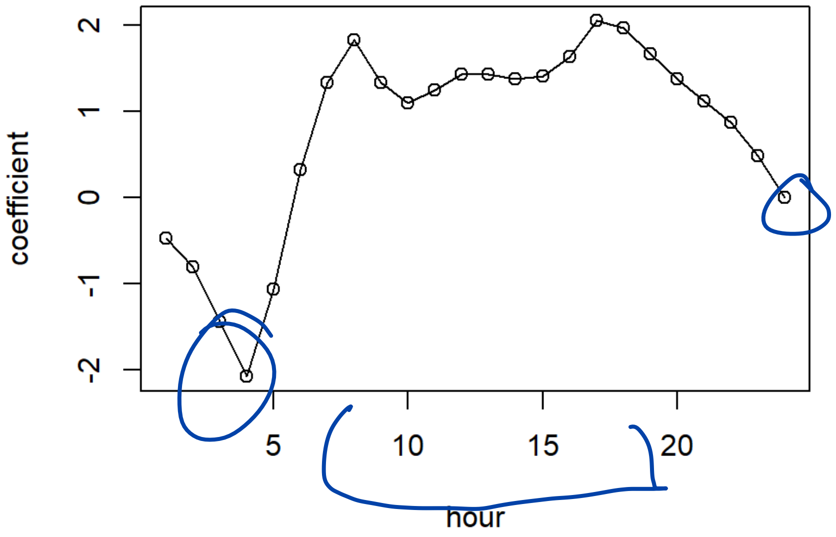
```
Call:
glm(formula = bikers ~ workingday + temp + weathersit + mnth +
    hr, family = poisson(), data = ISLR2::Bikeshare)

Coefficients:
                           Estimate Std. Error  z value Pr(>|z|)
(Intercept)                2.693688   0.009720  277.124  < 2e-16 ***
workingday                 0.014665   0.001955    7.502 6.27e-14 ***
temp                       0.785292   0.011475   68.434  < 2e-16 ***
weathersitcloudy/misty    -0.075231   0.002179  -34.528  < 2e-16 ***
weathersitlight rain/snow -0.575800   0.004058 -141.905  < 2e-16 ***
weathersitheavy rain/snow -0.926287   0.166782   -5.554 2.79e-08 ***
mnthFeb                    0.226046   0.006951   32.521  < 2e-16 ***
mnthMarch                  0.376437   0.006691   56.263  < 2e-16 ***
mnthApril                  0.691693   0.006987   98.996  < 2e-16 ***
mnthMay                    0.910641   0.007436  122.469  < 2e-16 ***
mnthJune                   0.893405   0.008242  108.402  < 2e-16 ***
mnthJuly                   0.773787   0.008806   87.874  < 2e-16 ***
mnthAug                    0.821341   0.008332   98.573  < 2e-16 ***
mnthSept                   0.903663   0.007621  118.578  < 2e-16 ***
mnthOct                    0.937743   0.006744  139.054  < 2e-16 ***
```

# Poisson regression - `Bikeshare` dataset

```r
plot(x = 1:12, y = c(0, glmBikeshare$coefficients[7:17]), type = 'o',
     xlab = "month", ylab = "coefficient", xaxt = "n")
axis(1, at=1:12, labels=substr(month.name, 1, 1))
plot(x = 1:24, y = c(glmBikeshare$coefficients[18:40], 0), type = 'o',
     xlab = "hour", ylab = "coefficient")
```



## Lecture Outline

- An overview of classification
- Logistic regression
- Poisson regression
- **Generalised linear models**

# Generalised linear models

| | Linear Regression | Logistic Regression | Poisson Regression | Generalised Linear Models |
|---|---|---|---|---|
| **Type of Data** | Continuous | Binary (Categorical) | Count | Flexible |
| **Use** | Prediction of continuous variables | Classification | Prediction of the number of events | Flexible |
| **Distribution of Y** | Normal | Bernoulli (Binomial for multiple trials) | Poisson | Exponential Family |
| $\mathbb{E}[Y \mid X]$ | $X\beta$ | $\frac{e^{X\beta}}{1+e^{X\beta}}$ | $e^{X\beta}$ | $g^{-1}(X\beta)$ |
| **Link Function Name** | Identity | Logit | Log | Depends on the choice of distribution |
| **Link Function Expression** | $\eta(\mu) = \mu$ | $\eta(\mu) = \log\left(\frac{\mu}{1-\mu}\right)$ | $\eta(\mu) = \log(\mu)$ | Depends on the choice of distribution |

GLM